Dual Number Subalgebras mapped to Digital Signal Processing Structures

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Higher-dimensional algebras in Digital Signal Processing

- Complex numbers in DSP: Essential
- Extension of concept (more than one imaginary component)?
- Sangwine et al.: Hypercomplex colour image processing
- Schütte 1990: “Reduced biquaternions” (RB) in DSP
  - RB-valued transfer functions
  - Efficient RB-valued multiplication and convolution
- Petrovsky, Parfieniuk, et al. 1999:
  - Novel (paraunitary) filter banks based on quaternions
- Extended concept: Filter banks based on RBs
Example algebra: Reduced biquaternions

- Reduced biquaternions: 4-dimensional commutative algebra

\[ a = a_1 + a_2 i + a_3 j + a_4 k, \quad a_1, a_2, a_3, a_4 \in \mathbb{R}, \]

\[ i^2 = j^2 = -1, \quad k^2 = +1, \quad ij = ji = k, \quad \ldots \]

- RB multiplication: 16 real multiplications and 12 real additions

- RBs comprise divisors of zero:
  Not every \( a \) is invertible, product \( ab \) may yield zero even if \( a \) and \( b \) are not zero, e.g.

\[ (1 + k)(1 - k) = 1 - k^2 = 1 - 1 = 0 \]

- Applying a special decomposition, the computational load for multiplication and convolution can be reduced to the half
Application: 4-channel analysis filter bank based on RBs

- Cascade of RB-valued first order allpasses
- Arbitrary individual channel bandwidths
- Paraunitary (perfect reconstruction)
- Low expenditure, group delay
Motivation: Raised questions

1. Example application is based on algebra exhibiting zero divisors
   - What does this mean for digital LTI systems?
2. Example application is processing real-valued signals
   - How can this be efficient?
Relation to modern mathematics

• Benjamin Peirce 1870:
  • Idempotents (divisors of zero): $e^2 = e$
  • (Right) Peirce decomposition: Algebra element $a$ represented as direct sum:
    \[ a = eb_1 + (1 - e)b_2, \quad a \in A, \quad b_1, b_2 \in B \]

• Subalgebras: $B \subset A$
• Nilpotents (divisors of zero): $n^p = 0, \ p > 1$

• Maclagan Wedderburn 1908:
  • Semi-simple algebra: Decomposable with Peirce decomposition
  • Radical $N$: Containing all nilpotent subalgebras
  • Any (associative) algebra is the sum of its radical $N$ and a semi-simple algebra.
Choosing comprehensible algebra examples

- Should exhibit higher-dimensional algebras’ properties in question
- Should be as simple as possible (low dimension)
- Two examples with dimension 2 (!):
  - Double numbers \( a = a' + a'' \omega, \quad \omega^2 = 1 \)
  - Dual numbers \( a = a' + a'' \varepsilon, \quad \varepsilon^2 = 0 \)
- Furthermore, they frequently emerge as subalgebras
Generalised complex numbers

- Three alternatives for imaginary unit $\gamma$:

\[ a = a' + a'' \gamma, \quad a', a'' \in \mathbb{R} \]

- $\gamma^2 = i^2 = -1$: Common complex numbers, division algebra
- $\gamma^2 = \omega^2 = +1$: Double numbers, comprising non-trivial idempotents
- $\gamma^2 = \varepsilon^2 = 0$: Dual numbers, comprising non-trivial nilpotents

- Multiplication (commutative):

\[ ab = a' a'' + (a' b'' + a'' b') \gamma + a'' b'' \gamma^2 \]

- Conjugate: $\bar{a} = a' - a'' \gamma$

- Semi-norm:

\[ N(a) = a \bar{a} = a'^2 - a''^2 \gamma^2 \in \mathbb{R} \]

- Zero divisors: $N(a) = 0$ except $a = 0$
### Zero divisors of generalised complex numbers

<table>
<thead>
<tr>
<th></th>
<th>Complex (i^2 = -1)</th>
<th>Double (\omega^2 = 1)</th>
<th>Dual (\varepsilon^2 = 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Idempotents</td>
<td>(e = 0, 1)</td>
<td>(e = 0, \frac{1+\omega}{2}, \frac{1-\omega}{2}, 1)</td>
<td>(e = 0, 1)</td>
</tr>
<tr>
<td>Nilpotents</td>
<td>(n = 0)</td>
<td>(n = 0)</td>
<td>(n = a''\varepsilon)</td>
</tr>
<tr>
<td>Semi-norm</td>
<td>(N(a) = a'^2 + a''^2)</td>
<td>(N(a) = a'^2 - a''^2)</td>
<td>(N(a) = a'^2)</td>
</tr>
</tbody>
</table>

e.g. \(\left(\frac{1+\omega}{2}\right)^2 = \frac{1+2\omega+\omega^2}{4} = \frac{2+2\omega}{4} = \frac{1+\omega}{2}\)
Double numbers: Peirce decomposition

- Peirce decomposition based on double numbers’ idempotents
  \[ a = \tilde{a}_1 \frac{1 + \omega}{2} + \tilde{a}_2 \frac{1 - \omega}{2}, \quad \tilde{a}_1, \tilde{a}_2 \in \mathbb{R} \]

- Orthogonal components:
  \[ \tilde{a}_1 = a' + a'', \quad \tilde{a}_2 = a' - a'' \]

- All operations can be performed component-wise, for instance multiplication and convolution
DSP structures resulting from subalgebras

- Structures determined by underlying algebra’s multiplication:
  - Generalised complex LTI system
  - Common complex LTI system
  - Double LTI system
  - Dual LTI system

- Impulse response: $h(k) = h'(k) + h''(k)\gamma$
- Generalised z-Transform:
  \[ H(z) = \mathcal{Z}\{h(k)\} = \sum_{k=-\infty}^{\infty} h'(k)z^{-k} + \sum_{k=-\infty}^{\infty} h''(k)z^{-k}\gamma \]
- Convolution theorem valid for all cases:
  \[ H(z)X(z) = \mathcal{Z}\{h(k) * x(k)\} \]
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Zero divisors in double transfer functions

- Let the double transfer function $H(z)$ have a zero divisor value at the distinct frequency $z_e$:
  \[ N[H(z_e)] = 0, \text{ e.g. } H''(z_e) = H'(z_e) = H(z_e) \]
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Double LTI system: $\Rightarrow$ Information loss

Common complex LTI system: $\Rightarrow$ no information loss
Zero divisors in dual transfer functions

- Let the dual transfer function $H(z)$ have a zero divisor value at the distinct frequency $z_e$:
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Dual LTI system:

Common complex LTI system:
Let the dual transfer function $H(z)$ have a zero divisor value at the distinct frequency $z_e$:

$$N[H(z_e)] = 0, \quad H'(z_e) = 0$$

**Dual LTI system:**

⇒ Information loss

**Common complex LTI system:**

⇒ no information loss
Processing of real signals with double system (1/2)

- No meaningful double signal known
- Processing of real input signal with double-valued non-recursive (FIR) system:

  ![Diagram showing processing of real signals with double system](image)

  - System is reduced to 2 used subsystems
• Cascaded double-valued FIR systems
  ⇒ From second stage on: Full usage of structure
• Double-valued IIR systems
  ⇒ Full usage of structure

Full usage of structure:
• 4 real subsystems involved in the entire transfer functions
• Alternative representation (due to Peirce decomposition):
  4 real subsystems with the computational load of 2 real subsystems
Processing chain exploiting Peirce decomposition

- Processing chain employing alternative representation:

- System $\tilde{H}(z)$ processing orthogonal components
- Doubling of real subsystems’ degree
- Only 2 independent subsystems, operating in parallel: Half expenditure
- Structure similar to common coupled (allpass) structures
Summary

- Double numbers (subalgebra):
  - Comprehensible example for algebra comprising idempotents
  - Peirce decomposition $\Rightarrow$ Efficient computation
- Dual numbers (subalgebra):
  - Comprehensible example for algebra comprising nilpotents
- Zero divisors (both idempotents and nilpotents) in transfer functions are singularities, comparable to zeros
- Processing of real signals feasible:
  - Dual numbers LTI systems far from being recommended as a self-contained system class
  - Double number LTI systems more promising, due to efficiency
Dual number LTI system of first order

Transfer function:
\[ H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} \]

- Coefficients \( a_1, b_0, b_1 \) are dual numbers
- Transfer function of first subsystem:
\[
H'(z) = \frac{b'_0 + \left[ b'_1 + a'_1 b'_0 \right] z^{-1} + a'_1 b'_1 z^{-2}}{1 + 2 a'_1 z^{-1} + a'^2_1 z^{-2}}
\]
- Transfer function of second subsystem:
\[
H''(z) = \frac{b''_0 + \left[ b''_1 + a''_1 b''_0 - a''_1 b'_0 \right] z^{-1} + \left[ a'_1 b''_1 - a''_1 b'_1 \right] z^{-2}}{1 + 2 a'_1 z^{-1} + a'^2_1 z^{-2}}
\]