Dual Number Subalgebras mapped to Digital Signal Processing Structures

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Abstract

Recently, applications of higher-dimensional (hypercomplex) algebras (e.g. quaternions, Clifford algebras) to Digital Signal Processing (DSP) emerge. Some of these employed algebras comprise idempotent and nilpotent elements. Regarding the latter, the consequences for DSP applications are addressed for the first time. As a simple example, we analyse digital LTI systems operating with dual numbers $a = a' + a'' \varepsilon$, $\varepsilon^2 = 0$, occurring as a subalgebra in many higher-dimensional algebras. We describe the general internal structure of such systems and their behaviour.

1. Introduction

The use of complex numbers in Digital Signal Processing (DSP) is essential, particularly because they allow for efficient sampling, modulation and baseband processing. However, it is natural to call for an extension to this successful concept, regarding the vast diversity of algebras known in mathematics. As far as we know, SCHÜTTE (1990) was the first to employ a non-division algebra for DSP [1]. Since then, the application of higher-dimensional algebras to DSP has made a promising progress, e.g. in the field of (colour) image [2] and vector-sensor array [3] processing. Recently, attempts have been made to define a hypercomplex analytic signal of a complex signal [4].

Some of these approaches (e.g. [2, 3]) utilise algebras which exhibit zero divisors: For such a number $a \neq 0$, no inverse is defined and a product $ab$ can vanish even if $a, b \neq 0$ [6]. Consequently, several questions of signal interpretation and validity of arithmetics are posed. Zero divisors can be a linear combination of idempotent ($\varepsilon^2 = \varepsilon$) or nilpotent ($\eta^2 = 0$) elements of an algebra. Therefore, we encounter two “types” of zero divisors. The impact of idempotent elements on DSP applications has been discussed in [5] on the basis of double (hyperbolic) numbers (complex numbers with imaginary unit $j$, where $j^2 = +1$). It has emerged that with a PEIRCE decomposition [6], the computational complexity with double numbers is reduced considerably. The objective of this contribution is to investigate the consequences of the existence of nilpotent elements in algebras used for DSP. Tracing back to WEDDERBURN (1908) [6, 7], it is possible to extract the nilpotent subalgebra (radical) of an associative higher-dimensional algebra. Assuming that the algebra is decomposed in its subalgebras, we can analyse them separately. A simple subalgebra example comprising nilpotent elements are the dual numbers (complex numbers with imaginary unit $\varepsilon$, where $\varepsilon^2 = 0$), which we choose for investigation.

In the following, we shortly review the alternatives of complex numbers (sec. 2), describe the inherent DSP structures of dual numbers, give a brief system theory of dual-valued transfer functions, plot a dual system of first order (sec. 3) and conclude the discussion (sec. 4).
2. Complex, double and dual numbers

We define a generalised complex number [8], comprising the imaginary unit \( \gamma \),
\[
\mathbf{a} = \mathbf{a} + \mathbf{a} \gamma, \quad \mathbf{a}, \mathbf{a} \in \mathbb{R}, \quad \gamma \notin \mathbb{R},
\]  
(2.1)
limited to three cases: The ordinary complex numbers \( \mathbb{C} \) with \( \gamma^2 = 1^2 = -1 \), the double (hyperbolic) numbers \( \mathbb{D} \) with \( \gamma^2 = j^2 = 1 \), and the dual numbers \( \mathbb{E} \) with \( \gamma^2 = \varepsilon^2 = 0 \).

In modern mathematical terms, these algebras form a commutative \( (ab = ba) \) ring with unity over \( \mathbb{R} \). Therefore their multiplication is distributive over addition, associative and commutative, deduced from (2.1):
\[
\mathbf{ab} = (\mathbf{a} + \mathbf{a} \gamma) (\mathbf{b} + \mathbf{b} \gamma) = \mathbf{a} \mathbf{b} + (\mathbf{a} \mathbf{b} + \mathbf{a} \gamma) \gamma + \mathbf{a} \gamma \mathbf{b} \gamma^2.
\]  
(2.2)

Obviously, a complex and double number multiplication consists of four real multiplications and two additions, whereas a dual number multiplication comprises only three real multiplications and one addition. However, it should be noted that for double numbers with a Peirce decomposition, the multiplication expense is virtually halved [1, 5].

Division (or element inversion) is carried out by
\[
\frac{\mathbf{b}}{\mathbf{a}} = \frac{\mathbf{ba}}{N(\mathbf{a})}, \quad N(\mathbf{a}) = \mathbf{a} \mathbf{a} = \text{det} (\mathbf{Ma}) = a' a'' - a'' a' \gamma^2 \in \mathbb{R},
\]  
(2.3)

where the seminorm \( N(\mathbf{a}) \) and the conjugate according to
\[
\mathbf{a} = \mathbf{a}' - \mathbf{a}'' \gamma, \quad \overline{\mathbf{a}} + \overline{\mathbf{b}} = \mathbf{a} + \mathbf{b}, \quad \mathbf{a} + \overline{\mathbf{a}} = 2 \mathbf{a}', \quad \mathbf{a} - \overline{\mathbf{a}} = 2 \mathbf{a}'' \gamma.
\]  
(2.4)
is used. In contrast to a norm, \( N(\mathbf{a}) \) can be zero even if \( \mathbf{a} \) is not zero, namely for double and dual numbers in case of \( \gamma^2 \geq 0 \) in (2.3); this does not apply for (ordinary) complex numbers. An element \( \mathbf{a} \) of these algebras with vanishing seminorm \( N(\mathbf{a}) \) is a zero divisor [6], having no inverse.

For dual numbers, the seminorm \( N(\mathbf{a}) = a'^2 \) is zero if and only if \( a' \) is zero. Therefore every dual number \( \mathbf{a} = \mathbf{a}' \varepsilon, \mathbf{a}'' \in \mathbb{R} \) is a zero divisor. Next, we determine the idempotent elements in \( \mathbb{E} [5]: e^2 = e \Rightarrow a'^2 + 2a'a'' \varepsilon = a' + a'' \varepsilon \Rightarrow a'^2 = a' \wedge 2a'a'' = a'' \Rightarrow a' = \{0, 1\} \wedge a'' = 0, \) which results in \( e = \{0, 1\} \) (no zero divisors).

Therefore, in contrast to double numbers, for dual numbers there is no decomposition leading to reduced computational complexity. Similarly, we seek after nilpotent elements: \( n^2 = 0 \Rightarrow a'^2 + 2a'a'' \varepsilon = 0 \Rightarrow a'^2 = 0 \wedge 2a'a'' = 0 \Rightarrow a' = 0, a'' \in \mathbb{R} \), which turn out to be equal to the set of zero divisors in \( \mathbb{E}: n = a' \varepsilon, a'' \in \mathbb{R} \).

3. Dual digital LTI systems

The output signal \( \mathbf{y}(k) \in \mathbb{E} \) of an assumed digital linear and time-invariant (LTI) system with impulse response \( \mathbf{h}(k) = \mathbf{h}'(k) + \mathbf{h}''(k) \varepsilon \in \mathbb{E} \), given the input signal \( \mathbf{x}(k) \in \mathbb{E} \), is carried out by dual convolution
\[
\mathbf{y}(k) = \mathbf{h}(k) * \mathbf{x}(k) = \mathbf{h}'(k) * \mathbf{x}'(k) + \left[ \mathbf{h}''(k) * \mathbf{x}'(k) + \mathbf{h}'(k) * \mathbf{x}''(k) \right] \varepsilon,
\]  
(3.1)
derived from the multiplication rule (2.2), implying associativity and commutativity of dual systems. From (3.1), the structure of such a system is determined: It consists of two distinct real subsystems with the impulse responses \( h''(k) \) and \( h''(k) \), the former emerging twice (fig. 1).

For a spectral representation, we need a suitable transform which, for dual numbers, is difficult to define, if an analogy to the frequency domain of real and complex signals and
systems is intended. As a remedy, we apply the (complex) $z$-Transform component-wise (due to linearity property) and get a dual transfer function:

$$H(z) = \mathcal{Z}\{h(k)\} = \sum_{k=-\infty}^{\infty} h'(k)z^{-k} + \sum_{k=-\infty}^{\infty} h''(k)z^{-k} \in \mathbb{C} \otimes E$$  \hspace{1cm} (3.2)

As indicated in (3.2), the algebra of $H(z)$ is a combination of complex and dual numbers, as the transfer functions $H'(z)$ and $H''(z)$ of the real subsystems are complex. They can also be obtained from $H(z)$, applying (2.4):

$$H'(z) = \frac{H(z) + \overline{H}(z)}{2}, \quad H''(z) = \frac{H(z) - \overline{H}(z)}{2},$$

whereas $\overline{\{\cdot\}}$ indicates a dual conjugation (2.4), not affecting the complex number $z$. The dual system in fig. 1 can also be represented as real $2 \times 2$ MIMO system with vectorial input $x(k)$ and output $y(k)$, and its transfer matrix $H(z)$:

$$Y(z) = H(z)X(z), \quad H(z) = \begin{bmatrix} H'(z) & 0 \\ H''(z) & H'(z) \end{bmatrix}, \quad X, Y(z) = \begin{bmatrix} \mathcal{Z}\{x, y'(k)\} \\ \mathcal{Z}\{x, y''(k)\} \end{bmatrix}.$$  \hspace{1cm} (3.4)

3.1. Nilpotent elements in dual signals and systems

Since for the time being, no meaningful interpretation of a dual signal $x(k) \in E$, a nilpotent sample $x(k_n)$, $n \in \mathbb{Z}$ of this signal just means that the real component $x'(k_n) = 0$ is zero. However, following the seminorm in (2.3), the signal power $\mathcal{N}(x(k_n)) = 0$ of such a sample also vanishes, regardless of the dual imaginary part $x''(k)$. Therefore the absolute value (norm) $x^2(k) + x''(k)$ is necessary to determine the signal power.

Having nilpotent coefficients in a dual system means that the real subsystem $H''(z)$ in (3.4) vanishes. However, even if every single coefficient is not a zero divisor, zero divisors can occur at distinct frequency points $\Omega_n$ of a dual transfer function, substituting $z := e^{j\Omega_n}$: $H^2(e^{j\Omega_n}) = 0$. This is the case at those $\Omega_n$, where the real subsystem $H'(z)$ has a zero. As can be seen from (3.4) and fig. 1, this means a data loss regarding the dual imaginary input component at that frequency point. This is very similar to zero divisor frequency points in double-valued systems [5] and even comparable with ordinary zeros in real or complex systems, completely annihilating the input signal at that frequency.

3.2. Recursive dual system of first order

Next, we briefly investigate a recursive dual LTI system of first order (fig. 2), being represented by the difference equation and dual transfer function

$$y(k) + a_1 y(k-1) = b_0 x(k) + b_1 x(k-1), \quad H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}, \quad a_1, b_0, b_1 \in E.$$  \hspace{1cm} (3.5)
with each coefficient a dual number. Alternatively, the system can be regarded as a composition of real subsystems $H(z)$ and $H''(z)$ (fig. 1): We apply (3.3), $\overline{a\overline{b} + \overline{\alpha}b} = 2a\overline{b}$ and $\overline{a\overline{b} - \overline{\alpha}b} = 2\left(a\overline{b} - a\overline{\alpha} + \overline{\alpha}b\right)\epsilon$ to (3.5) and get:

$$H'(z) = \frac{1}{2} \left(\frac{b_0 + b_1z^{-1}}{1 + a_1z^{-1}} + \frac{\overline{b}_0 + \overline{b}_1z^{-1}}{1 + \overline{a}_1z^{-1}}\right) = \frac{b_0' + \left[b_1' + a_1'\overline{b}_0'\right]z^{-1} + a_1'\overline{b}_1'z^{-2}}{1 + 2a_1'z^{-1} + \overline{a}_1'z^{-2}}. \quad (3.6)$$

$$H''(z) = \frac{b_0'' + \left[b_1'' + a_1''\overline{b}_0'' - a_1''\overline{b}_1''\right]z^{-1} + \left[a_1''\overline{b}_1'' - a_1''\overline{b}_1''\right]z^{-2}}{1 + 2a_1'z^{-1} + \overline{a}_1'z^{-2}}. \quad (3.7)$$

We see that the transfer functions (3.6) and (3.7) of its real subsystems are of second order. This means the doubling of the system degree, compared to a real system, known also from recursive complex systems. For both complex, dual (and also double [5]) systems, this is only the case if the recursive coefficients (here: $a_1$) do not vanish. However, there is a remedy for FIR systems to use cascades in order to employ the full structure [5]. In contrast to the subsystems’ transfer functions of complex (and double [5]) systems, (3.6) and (3.7) only depend on the real parts of the coefficients, $a_1$, $b_0$ and $b_1$. Moreover, the poles $z_{\infty} = 1/a_1$ of both (3.6) and (3.7) are real and double, also limiting the degrees of freedom for feasible transfer functions.

### 4. Conclusion

A self-contained use of dual numbers in DSP applications is far from being recommended. This is due to both the reduced transfer function flexibility and the lack of an efficient multiplication. However, as subalgebras, they emerge in significant applications of higher-dimensional algebras in DSP. With the aid of modern mathematical knowledge, it should be possible to extract dual number subalgebras from these algebras, if nilpotent elements emerge. Here, they have served as a simple object of investigation. Regarding nilpotent elements, we have shown that the impact on DSP is comparable with that of other zero divisors, like idempotent elements. It is subject to future investigations to transfer this observation to higher-dimensional algebras.

### REFERENCES