Linear Phase FIR Filter Design for Multirate Systems based on Fast Convolution

Alexandra Groth, Frank Budke, Heinz G. Göckler, Gennaro Evangelista
Digital Signal Processing Group, Ruhr-Universität Bochum,
Universitätssstr. 150, D-44780 Bochum, Germany
Tel: +49 234 3222805; fax: +49 234 3214100
email: groth@nt.ruhr-uni-bochum.de

Abstract

Fast convolution is well known for efficient FIR filtering. In a recently proposed approach [2] it is extended to multirate signal processing. Thereby additional savings in computational expenditure can be obtained by suitable frequency domain filter coefficients. Hence, new design methods for linear phase interpolation and decimation filters in multirate systems with fast convolution are suggested.

1 Introduction

In digital systems fractional sample rate conversion (FSRC) [1] is one of the basic tasks (Fig.1). In order to reduce the computational expenditure various techniques have been investigated. One of them [2] comprises the extension of fast convolution, well known from efficient filtering algorithms, to FSRC.

![Figure 1: Fractional sample rate conversion](image)

Figure 2: Efficient fractional sample rate conversion with fast convolution

FFT, an MIMO (Multiple Input Multiple Output) system including up- and downsampling and an IFFT. In doing so, the required MIMO system (Fig.3) comprises upsampling by an L-fold repetition of the input spectrum and downsampling by adding every \( \frac{1}{L} \)th multiplication result.

![Figure 3: K × L MIMO system](image)

The resulting overall system (Fig.2) consists of an

\* This work was supported by Deutsche Forschungsgemeinschaft under contract GO 849/1-1

A reduction in computational expenditure beyond the already achieved savings can only be obtained if the frequency domain zero phase filter coefficients \( H(l) \) become zero, one, powers-of-two, or if \( |H(m \frac{L}{M} K)| \) with \( m = 0, \ldots, M-1 \) are chosen to be equal so that they can be shifted behind the summation point. These restrictions are frequently postulated for time domain filter coefficients, but are not adequately investigated for frequency domain coefficients.
In the following section filter design methods are presented, whereby the desired frequency response is optimally approximated by exploiting all available degrees of freedom.

2 New Filter Design Methods

In Fig.4 an example of a desired frequency response $D_0(e^{j\Omega})$ of the zero phase filter and an example of a sample frequency grid is depicted. This grid represents the $L K$ equidistant points at which the approximated frequency response will be sampled in order to obtain the frequency domain filter coefficients $H(l)$.

![Desired zero phase filter](Image)

Figure 4: Desired zero phase filter $D_0(e^{j\Omega})$ with marked sample frequency grid of $H(e^{j\Omega})$

In the following the filter length is assumed to be $N$. Hence, $\frac{N+1}{2}$ degrees of freedom are at disposal for the filter design. Each of the previously mentioned conditions of $H(l)$ utilizes either one or two degrees of freedom. But because of the frequency response symmetry at most $N$ multiplications per input sample can be saved if, for instance, $H(l) \in \{0, 1, 2^{-i} | i \in \mathbb{N}\}$.

Due to the necessity to force the cyclic convolution to perform a linear convolution we always have $N < L K$, the overall number of samples of $H(e^{j\Omega})$. That is why still non trivial multiplications remain.

As a result, every design method has to consider two aspects: i) The algorithm has to select at most $\frac{N+1}{2}$ suitable sample frequencies out of the available ones, where ii) convenient desired values $H(l)$ have to be assigned. In addition the overall frequency response has to be approximated best. In the following, two methods applying different approximation criteria are proposed.

2.1 Frequency-Sampling Design

The frequency-sampling design is a straightforward design method, where $\frac{N+1}{2}$ samples of the desired frequency response $D_0(e^{j\Omega})$ are used to find the appropriate $\frac{N+1}{2}$ time domain filter coefficients by simply solving a completely determined system of linear equations [3]. Preferably frequency samples are used where the desired values, prescribed on the sample frequency grid, are trivial to implement (Fig.5). The choice of those $\frac{N+1}{2}$ out of $\frac{L K+1}{2}$ suitable sample frequencies is governed by a knowledge-based trial and error procedure.

![Possible choice of frequency samples](Image)

Figure 5: Possible choice of frequency samples for the frequency-sampling design ($N = 21$)

Note that the resulting frequency response attains the desired values exactly at the set of sampling frequencies being, in general, not equidistantly distributed over $[0, \pi]$. The overall set $\{H(l) | l = 0, \ldots, L K - 1\}$ of frequency domain filter coefficients is obtained by extending $h(m)$ to length $L K$ by zero padding and subsequent calculation of $\text{FFT}\{h(m)\}$.

This design method implies three drawbacks: i) The frequency response approximation is not related to a meaningful approximation criterion. ii) The overall frequency response can only be controlled by suitable allocation of the sample frequencies on the sample frequency grid. iii) A reduction of computational expenditure is only possible, if the desired function $D_0(e^{j\Omega})$ consists of suitable values at the sample frequencies.

2.2 Least Squared Error Frequency-Domain Design

In the following we regard linear phase FIR filter design with a squared error approximation criterion [3]. Thereby the squares of the error in the frequency domain are measured and minimized at a finite set of sample frequency points $\{\Omega_l | l = 0, \ldots, L - 1\}$. As a result, in contrast to the preceding design, we control the frequency response by a number of sample points noticeably larger than the order of the filter. Obviously $\{\Omega_l | l = 0, \ldots, L - 1\}$ need not comprise $\frac{L K+1}{2}$.

In order to obtain a frequency response which includes up to $N$ trivial multiplications on the sample frequencies grid, an iterative procedure is developed.

**Step1: Initialization**

To start with, design a linear phase FIR filter
by minimizing the squared error \( \|e\|^2 = e^T e \) on \( \{\Omega, \lambda = 0, ..., \Lambda - 1\} \) [3]. Remember that \( e \) is given by

\[
F \cdot h = D_0 + e .
\]

Obtain the LMSE time domain filter coefficients

\[
h = (F^T F)^{-1} F^T D_0 .
\]

**Step 2: Selection**

Calculate the frequency samples \( \{H(l)|l = 0, ..., LK - 1\} \). Set, according to a suitable selection criterion, at least one of those frequency samples to \( H_Q(l) \in \{0, 1, 2^{-1}l \in \mathbb{N} \text{ or multiplicative to equal values}\}. Note that the associated multiplications turn trivial reducing the computational expenditure of the system.

**Step 3: Redesign**

Redesign the LMSE time domain filter coefficients subject to the fact that the subset of \( N_f \) quantized frequency domain filter coefficients \( H_Q = \{H_Q(l)\} \) is retained. To this end, set up a new system of equations resulting from predefinition of those desired values \( H_Q \)

\[
Th = T_1 a_1 + T_2 a_2 = H_Q ,
\]

resp.

\[
a_1 = T_1^{-1} [H_Q - T_2 a_2] ,
\]

where Eq. (3) is rearranged by subdividing \( h \) into \( a_1 = [h(0), ..., h(N_f - 1)] \) and \( a_2 = [h(N_f), ..., h(N_f - 1)] \). Thereby \( a_1 \) represents the \( N_f \) exploited degrees of freedom which can unambiguously be calculated if the remaining filter coefficients \( a_2 \) are known. Apply the same procedure to the system of equations (Eq. (1)) describing the LMSE approximation problem

\[
Fh = F_1 a_1 + F_2 a_2 = D_0 + e .
\]

Subsequently, eliminate \( a_1 \) by Eq. (3) and compute

\[
a_2 = \left[ (F_1 T_1^{-1} T_2 - F_2)^T (F_1 T_1^{-1} T_2 - F_2) \right]^{-1}
\]

\[
(F_1 T_1^{-1} T_2 - F_2)^T (F_1 T_1^{-1} H_Q - D_0) .
\]

Finally, determine the remaining unknown filter coefficients \( a_1 \) by solving Eq. (3).

Repeat steps 2 and 3 iteratively. Terminate the iteration if i) all degrees of freedom are spent, or if ii) the redesign with the reduced number of degrees of freedom leads to an unacceptable deterioration of the overall frequency response. In the latter case delete the result of the last iteration.

**3 Examples**

The amount of computations (complex multiplications per output sample) saved by the proposed approaches is best illustrated by an example. Assume the digital system to be an eight channel, analysis filter bank with a decimation factor \( M = 8 \), with essential arbitrary spaced channel frequencies and all channels sharing the same bandwidth. For that purpose a prototype filter with a passband edge frequency \( f_p = \pi / 8 \), a stopband edge frequency \( f_s = \pi / 4 \) and a quasi continuous squared error \( \|e\|_2 = 10^{-3} \) is required.

On these conditions a standard least squared error (LMSE) design [3] leads to a filter of length \( N = 47 \) \( \|e\|_2 = 9.2 \cdot 10^{-4} \). Hence, a polyphase realization requires 47 complex multiplications per output sample.

**3.1 Standard LMSE Design**

In comparison, we regard a fast convolution implementation of this system as depicted in Fig. 7.

**Figure 6:** Result of the first iteration: One coefficient \( H(l) \) is found

Using an overlap and add algorithm in addition, the prototype filter has to be of length \( N = 8m + 1 \), \( m \in \mathbb{N} \). Hence, \( N \) is incremented \( (N = 49) \) and the filter is redesigned (conventional LMSE algorithm: \( \|e\|_2 = 6.5 \cdot 10^{-4} \)). However, attaining the optimal FFT block length \( K = 2^3 \), a minimization of the overall com-
putational expenditure

\[ C(\text{System}) = M \frac{1}{N} C(\text{FFT}) + \frac{1}{K} + C(\text{IFFT}) \]

is required. Thereby \( K \) is assumed to be power-of-two (IFFT length: \( K/8 \)) such that a split radix algorithm [4] with \( C(\text{FFT/IFFT}) = K/3 \log_2 K - K + 4/3 \) can be applied. Eventually, we obtain a computational expenditure \( C \) of 7.75 complex multiplications per output sample.

3.2 Frequency-Sampling Design

Now we turn towards the new approaches. Following the frequency-sampling design, a knowledge-based trial and error strategy for the setting of the frequency samples to \( H_0 \) is employed. Therefore, the desired frequency response \( D_0 \) is sampled equidistantly in the pass- and stopband. Note that \( 1 - 2^{-i} \) or \( 2^{-i} \), respectively, is assigned to cutoff frequency samples in compliance with the expected maximal deviation. To start the design procedure, we assume \( N = 49 \) and compute the optimal FFT block length to \( K = 2^8 \). Hence, we attain a prototype filter with a quasi continuous approximation error of \( ||e||_2 = 9.4 \cdot 10^{-3} > 10^{-3} \). In general, it can be observed that \( ||e||_2 \) seldom falls below this order of magnitude applying knowledge based trial and error strategies.

Only with an increased filter length \( N = 8m + 1 = 65 \) \((\Rightarrow K = 2^8, \text{edge: } 2^{-8})\) and a higher density of frequency samples in the stopband, a filter with a sufficient approximation error \( ||e||_2 = 4.2 \cdot 10^{-4} \) is obtained. As a result, 65 multiplications are trivial in each branch of the filter bank leading to 7.7% savings in computational expenditure compared to the standard LMSE design.

3.3 Modified LMSE Design

As aforementioned, a frequency sample design is not related to a meaningful approximation criterion. Hence, a modified LMSE frequency domain design seems to be a more promising approach. Thereby, a suitable criterion for the setting of frequency samples to \( H_0 \) appears to be the following: i) Choose the frequency sample with the value closest to \( \{0, 1, 2^{-i} | i \in N \} \) next to \( \pi \). ii) In the following iteration, make a rough estimation for next sample by assuming an equidistant location of the frequency samples. Search on either side of that point and choose the frequency sample causing the smallest error as consequence of assignment. iii) Find the other frequency sample by the same procedure.

Thus, we obtain a filter (Fig.8) with \( N = 49 \), \( ||e||_2 = 9.5 \cdot 10^{-4} \) and \( N_f = 46 \) trivial multiplications (0 or 1). Again, the optimal block length is computed \( (K = 2^8) \) and following from it, a reduction in computational expenditure of 10.7% compared to the standard LMSE design is attained.

![Figure 8: Attenuation resulting form a) standard LMSE (dashed line) and b) modified LMSE approach with 46 trivial multiplications (solid line)](image)

In most cases, it can be observed that the elimination of the last few degrees of freedom leads to a highly increased error \( ||e||_2 \). Hence, the modified LMSE algorithm produces an error only slightly higher than the standard approach if a few (here: 3) degrees of freedom are left for the minimization of \( ||e||_2 \) on the remaining frequency samples \( \Omega_0 \).

Note that all modified filter design techniques are most suitable for systems with small FFT/IFFT block lengths and small ratios \( LK/N \). Here, the computational complexity \( C \) is primarily determined by the savings due to trivial multiplications.

4 Conclusions

Well known filter design methods like frequency-sampling and least squared error frequency-domain design are modified in order to obtain efficient filters for fast convolution in multirate systems. As a result, savings of up to 10% are obtained. Note that the designed filters are still suitable if segmentation methods, such as overlap and add, are introduced. In future, design methods will be extended to Chebyshev approximation.

References


